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ON THE PRODUCT OF TWO π-DECOMPOSABLE SOLUBLE GROUPS

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Abstract

Let the group $G = AB$ be a product of two π-decomposable subgroups $A = O_\pi(A) \times O_{\pi'}(A)$ and $B = O_\pi(B) \times O_{\pi'}(B)$ where π is a set of primes. The authors conjecture that $O_\pi(A)O_\pi(B) = O_\pi(B)O_\pi(A)$ if π is a set of odd primes. In this paper it is proved that the conjecture is true if A and B are soluble. A similar result with certain additional restrictions holds in the case $2 \in \pi$. Moreover, it is shown that the conjecture holds if $O_{\pi'}(A)$ and $O_{\pi'}(B)$ have coprime orders.

1. Notation and Preliminaries

All groups considered are finite.

The aim of this paper is to study groups $G = AB$ which are factorized as the product of π-*decomposable* subgroups A and B, for a set of primes π. A group X is said to be π-*decomposable* if $X = X_\pi \times X_{\pi'}$ is the direct product of a π-subgroup and a π'-subgroup, where π' stands for the complementary of π in the set of all prime numbers. Moreover, we always use X_π to denote a Hall π-subgroup of any group X.

More precisely we take further the study that was started in [**12**]. The main result in that paper states the following:

Theorem 1. *Let π be a set of odd primes. Let the group $G = AB$ be the product of a π-decomposable subgroup A and a π-subgroup B. Then $A_\pi = O_\pi(A) \le O_\pi(G)$.*

It is worth recalling the following result, which is Lemma 1 in [**12**] and provides an equivalent statement to this theorem.

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Lemma 1. *Let the group $G = AB$ be the product of a π-decomposable subgroup $A = A_\pi \times A_{\pi'}$ and a π-subgroup B. Then the following statements are equivalent:*

(i) *$A_\pi \leq O_\pi(G)$;*

(ii) *G contains Hall π-subgroups and $A_\pi B = BA_\pi$ is a Hall π-subgroup of G.*

The starting point for our work is the theorem of Kegel and Wielandt which states the solubility of a group which is the product of two nilpotent subgroups.

For the proof of this theorem Kegel found a very useful criterion for the non-simplicity of a finite group in terms of some suitable permutability conditions on subgroups ([**13**, Satz 3]). It was improved by Wielandt in [**15**, Satz 1]. (See also [**1**, Lemmas 2.4.1, 2.5.1].) We state here a reformulation of these results which is convenient for our purposes.

Lemma 2. *Let the group $G = AB$ be the product of the subgroups A and B and let A_0 and B_0 be normal subgroups of A and B, respectively. If $A_0B_0 = B_0A_0$, then $A_0^g B_0 = B_0 A_0^g$ for all $g \in G$.*

Assume in addition that A_0 and B_0 are π-groups for a set of primes π. If $O_\pi(G) = 1$, then $[A_0^G, B_0^G] = 1$.

(We note that this result is applicable in particular if $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$ are π-decomposable and considering $A_0 = A_\pi$ and $B_0 = B_\pi$.)

Proof: Let $g \in G$ and consider $g = ab$ with $a \in A$ and $b \in B$. Since A_0 and B_0 are normal subgroups of A and B, respectively, and they permute, we have:

$$A_0^g B_0 = A_0^{ab} B_0 = (A_0B_0)^b = (B_0A_0)^b = B_0 A_0^{ab} = B_0 A_0^g.$$

Now the final assertion follows from [**1**, Lemma 2.5.1]. □

If $G = AB$ is the product of nilpotent subgroups A and B, then the hypotheses of this result for $A_0 = A_p$ and $B_0 = B_p$, the Sylow p-subgroups of A and B, respectively, and for any prime p, hold. This fact is in the core of the solubility of the group G.

Our aim is to find a more general structure involving π-decomposable groups for which these hypotheses also hold. Then, together with Lemma 2, our results also provide non-simplicity criteria for a group G.

Precisely we conjecture the following:

Conjecture. *Let π be a set of odd primes. Let the group $G = AB$ be the product of two π-decomposable subgroups $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$. Then $A_\pi B_\pi = B_\pi A_\pi$ and this is a Hall π-subgroup of G.*

Theorem 1 provides already a first approach to this conjecture. We state next another case for which the conjecture holds and that follows from Theorem 1. For notation, we set $\pi(G)$ for the set of prime divisors of $|G|$, the order of the group G.

Proposition 1. *Let π be a set of odd primes. Let the group $G = AB$ be the product of two π-decomposable subgroups $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$. Assume in addition that $(|A_{\pi'}|, |B_{\pi'}|) = 1$. Then $A_\pi B_\pi = B_\pi A_\pi$.*

Proof: Since $2 \in \pi'$ and $(|A_{\pi'}|, |B_{\pi'}|) = 1$ we may assume w.l.o.g. that $2 \notin \pi(B)$. Now we consider the set of odd primes $\sigma := \pi(B) \cup \pi(A_\pi)$. Then G is the product of the σ-decomposable subgroup A and the σ-subgroup B. From Theorem 1 it follows that B and $A_\sigma = A_\pi$ permutes. Considering now the group BA_π, we can deduce that B_π permutes with A_π as desired. $\square$

It is worthwhile emphasizing that the conjectured result holds in the significant case when $(|A|, |B|) = 1$. In particular, our results extend previous ones of Berkovič [**4**], Arad and Chillag [**3**], Rowley [**14**] and Kazarin [**9**], where products of a 2-decomposable group and a group of odd order, with coprime orders, were considered.

In this paper we will study as a first step the structure of a minimal counterexample to our conjecture. Afterwards we will prove it under the additional hypotheses that A and B are soluble groups. In the case of soluble factors, we will consider also the analogous problem when π is a set of primes containing the prime 2. As a consequence of these results we deduce in Corollary 1 a criterion of π-separability for a group which is the product of π-decomposable soluble factors, for an arbitrary set of primes π.

First we state some more notation. If n is an integer and p a prime number, we denote by n_p the largest power of p dividing n. A group G satisfies the C_π-property if G possesses a unique conjugacy class of Hall π-subgroups. Moreover G satisfies the D_π-property if it satisfies the C_π-property and every π-subgroup of G is contained in some Hall π-subgroup of G. We recall that a π-separable group satisfies the D_π-property.

We need specifically the following result (see [**1**, Corollary 1.3.3]).

Lemma 3. *Let the group $G = AB$ be the product of the subgroups A and B. Then for each prime p there exist Sylow p-subgroups A_p of A and B_p of B such that A_pB_p is a Sylow p-subgroup of G.*

For products of soluble subgroups the following lemma will be also used.

Lemma 4. *Let $G = AB = AN = BN$ be a group with A and B soluble subgroups of G and with a unique minimal normal subgroup N, which is non-abelian. Let $N = N_1 \times \cdots \times N_r$ with $N_1 \cong N_i$ be a non-abelian simple group, $i = 1, \dots, r$. Then:*

(i) *A and B act transitively by conjugacy on the set $\Omega = \{N_1, \dots, N_r\}$ of direct factors of N. Moreover, $N \cap A = \times_{i=1}^{r}(N_i \cap A)$ and $N \cap B = \times_{i=1}^{r}(N_i \cap B)$.*

(ii) *$|N_1|$ divides $|\mathrm{Out}(N_1)||N_1 \cap A||N_1 \cap B|$.*

Proof: See Lemmas 2.3 and 2.5 of [**10**]. □

2. The minimal counterexample

Proposition 2. *Let π be a set of odd primes. Assume that the group $G = AB$ is the product of two π-decomposable subgroups $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$, and G is a counterexample of minimal order to the assertion $A_\pi B_\pi = B_\pi A_\pi$.*

Then G has a unique minimal normal subgroup $N = N_1 \times \cdots \times N_r$, which is a direct product of isomorphic non-abelian simple groups $N_1, \dots, N_r$. Moreover $G = AN = BN = AB$, $(|A_{\pi'}|, |B_{\pi'}|) \neq 1$ and $A_{\pi'} \cap B_{\pi'} = 1$.

Proof: First note that $A_\pi \neq 1$ and $B_\pi \neq 1$. Moreover, $|\pi(G) \cap \pi| > 1$, because of Lemma 3, and also $(|A_{\pi'}|, |B_{\pi'}|) \neq 1$ by Proposition 1; in particular, $A_{\pi'} \neq 1$ and $B_{\pi'} \neq 1$. We split the proof into the following steps:

Step 1. The group G has a unique minimal normal subgroup N, which is neither a π-group nor a π'-group. In particular, N is not soluble. Consequently, $N = N_1 \times \cdots \times N_r$ with $N_1 \cong N_i$ a non-abelian simple group, $i = 1, \dots, r$.

Let N be a minimal normal subgroup of G and assume that there exists $M \neq N$ another minimal normal subgroup of G. The choice of G implies that $A_\pi B_\pi N/N$ is a subgroup of G/N and $A_\pi B_\pi M/M$ is a subgroup of G/M. Then

$$O^\pi(\langle A_\pi, B_\pi \rangle) \leq N \cap M = 1.$$

This implies that $\langle A_\pi, B_\pi \rangle$ is a π-group and, consequently, $\langle A_\pi, B_\pi \rangle = A_\pi B_\pi$, a contradiction.

If N is a π-group, then $\langle A_\pi, B_\pi \rangle \leq A_\pi B_\pi N$ is a π-group which implies the contradiction $\langle A_\pi, B_\pi \rangle = A_\pi B_\pi$, as $|A_\pi B_\pi| = |G|_\pi$ is the largest π-number dividing $|G|$.

Assume now that N is a π'-group. Note that

$$|A_\pi(B_\pi N)| = \frac{|A_\pi||B_\pi||N|}{|A_\pi \cap B_\pi N|}$$

and so $|A_\pi B_\pi N/N|$ is a π-number. Consequently, $X := A_\pi B_\pi N$ is a π-separable group and, in particular, it satisfies the D_π-property. We deduce now that there exists a Hall π-subgroup X_π of X and an element $x \in X$ such that $A_\pi B_\pi^x \subseteq \langle A_\pi, B_\pi^x \rangle \leq X_\pi$. But $|A_\pi B_\pi^x| = |G|_\pi$ which implies in particular that $A_\pi B_\pi^x = X_\pi$ is a subgroup of G. Since $G = AB$ and A_π and B_π are normal subgroups of A and B respectively, it follows that $A_\pi B_\pi$ is a subgroup of G.

Put now $H = \langle A_\pi, B_\pi \rangle$. Then the following properties hold:

Step 2. $N \leq H \trianglelefteq G$.

From [**1**, Lemma 1.2.2] we have that $N_G(H) = N_A(H)N_B(H)$. If $N_G(H)$ is a proper subgroup of G, then $A_\pi B_\pi$ is a subgroup of G by the choice of G, which is a contradiction. Hence H is a normal subgroup of G and so $N \leq H$.

Step 3. $G = AH = BH = AB$.

Observe that $AH = A(AH \cap B)$. If AH is a proper subgroup of G, then the choice of G implies again the contradiction $A_\pi B_\pi = B_\pi A_\pi$. Therefore $G = AH$ and, analogously, $G = BH$.

Step 4. $H = A_\pi B_\pi N$.

This is clear since $A_\pi B_\pi N$ is a subgroup of G and $N \leq H \leq A_\pi B_\pi N \leq H$.

Step 5. $A_{\pi'} N = B_{\pi'} N = A_{\pi'} B_{\pi'} N$.

Since $G = AH = AB_\pi N$, we deduce that

$$\begin{aligned} B &= B_\pi(B \cap AN) = B_\pi((B_\pi \cap AN) \times (B_{\pi'} \cap AN)) \\ &= B_\pi(B_{\pi'} \cap AN) = B_\pi B_{\pi'}. \end{aligned}$$

Then $B_{\pi'} = B_{\pi'} \cap AN$, that is, $B_{\pi'} \leq AN$ and, consequently, $B_{\pi'} \leq A_{\pi'} N$.

Analogously the equality $G = BH = BA_\pi N$ implies that $A_{\pi'} \leq B_{\pi'} N$.

Therefore $A_{\pi'} N = B_{\pi'} N = A_{\pi'} B_{\pi'} N$.

Step 6. $G/N = O_{\pi'}(G/N) \times O_\pi(G/N)$.

Note first that $H/N = A_\pi B_\pi N/N \in \mathrm{Hall}_\pi(G/N)$ and $H/N \trianglelefteq G/N$. On the other hand, we deduce from Step 5 that $A_{\pi'} N/N = B_{\pi'} N/N$ is a Hall π'-subgroup of G/N normalized by AN/N and by BN/N, that is, it is normal in G/N, and the assertion follows.

Step 7. $A_{\pi'} \cap B_{\pi'} = 1$.

If $L = A_{\pi'} \cap B_{\pi'}$, then $N \leq \langle A_\pi, B_\pi \rangle \leq C_G(L)$, and so $L \leq C_G(N) = 1$.

Step 8. Assume that $1 \neq M \trianglelefteq G$ and $K := AM \neq G$. Then $O_\pi(K) = 1$, $A_\pi \tilde{B}_\pi \in \mathrm{Hall}_\pi(K)$ and $[A_\pi^K, \tilde{B}_\pi^K] = 1$, where $\tilde{B}_\pi := B_\pi \cap AM = B_\pi \cap A_\pi M$. Moreover, $\tilde{B}_\pi \neq 1$ and $B_\pi \cap M = \tilde{B}_\pi \cap M = 1$.

First observe that $[O_\pi(K), N] \leq O_\pi(K) \cap N = 1$, which implies $O_\pi(K) \leq C_G(N) = 1$. Moreover, since $K = AM = A(AM \cap B) < G$, the choice of G implies that $T := A_\pi \tilde{B}_\pi = \tilde{B}_\pi A_\pi \in \mathrm{Hall}_\pi(K)$. Hence, from Lemma 2, it follows that $[A_\pi^K, \tilde{B}_\pi^K] = 1$.

Suppose now that $\tilde{B}_\pi = 1$. Then $T = A_\pi \in \mathrm{Hall}_\pi(K)$ and $A_\pi \cap M \in \mathrm{Hall}_\pi(M)$. Note that $A_\pi \cap M \neq 1$ because otherwise M would be a π'-group, which contradicts Step 1. Since π is a set of odd primes, then M satisfies the C_π-property by [**8**, Theorem A] and so, by the Frattini argument, we conclude that $G = MN_G(A_\pi \cap M)$. Hence

$$|G : N_G(A_\pi \cap M)| = |M : N_M(A_\pi \cap M)|$$

is a π'-number, since $A_\pi \cap M \in \mathrm{Hall}_\pi(N_M(A_\pi \cap M))$, and so $|G|_\pi = |N_G(A_\pi \cap M)|_\pi$. Note also that $N_G(A_\pi \cap M) \neq G$, by Step 1. Then, by the choice of G, $N_G(A_\pi \cap M) = A(B_\pi \cap N_G(A_\pi \cap M)) \times (B_{\pi'} \cap N_G(A_\pi \cap M))$ satisfies the theorem, that is,

$$A_\pi(B_\pi \cap N_G(A_\pi \cap M)) \in \mathrm{Hall}_\pi(N_G(A_\pi \cap M)).$$

But $|A_\pi(B_\pi \cap N_G(A_\pi \cap M))| = |N_G(A_\pi \cap M)|_\pi = |G|_\pi = |A_\pi B_\pi|$ implies that $B_\pi \cap N_G(A_\pi \cap M) = B_\pi$ and so $A_\pi B_\pi$ is a subgroup, a contradiction. This proves that $\tilde{B}_\pi \neq 1$.

Finally note that $B_\pi \cap M = \tilde{B}_\pi \cap M$ is normalized by both B_π and A_π because $[A_\pi, \tilde{B}_\pi] = 1$. Hence $N \leq \langle A_\pi, B_\pi \rangle$ normalizes $B_\pi \cap M$ and so $[B_\pi \cap M, N] \leq B_\pi \cap M \cap N = B_\pi \cap N = 1$, since this is a π-group normalized by N. Therefore $B_\pi \cap M \leq C_G(N) = 1$ and the last assertion follows.

Step 9. A acts transitively on the set $\Omega = \{N_1, \ldots, N_r\}$.

Assume that this is not true and take $R := \cap_{i=1}^r N_G(N_i) \trianglelefteq G$. Then $AR < G$ and we can apply Step 8 with $M = R$. In particular, from the facts that $\tilde{B}_\pi = B_\pi \cap AR \neq 1$ and $B_\pi \cap R = \tilde{B}_\pi \cap R = 1$ we deduce that $\tilde{B}_\pi \nleq R$. Then there exists $1 \neq b \in \tilde{B}_\pi \setminus R$. Without loss of generality we may assume that $b \notin N_G(N_1)$, and so $|\Omega_{\langle b \rangle}(N_1)| \geq 2$, where $\Omega_{\langle b \rangle}(N_1)$ denotes the orbit of N_1 under the action of b on $\Omega = \{N_1, \ldots, N_r\}$. On the other hand, since $\tilde{B}_\pi \leq RA_\pi$, then $b = ca$ for some $c \in R$ and $a \in A_\pi$. Since R normalizes each N_i, we have $\Omega_{\langle b \rangle}(N_1) = \Omega_{\langle a \rangle}(N_1)$. Now note that $[N_1, \langle b \rangle] = N_{i_1} \times \cdots \times N_{i_k}$, where $\Omega_{\langle b \rangle}(N_1) = \{N_1 =$

$N_{i_1}, \dots, N_{i_k}\} \subseteq \Omega$. Analogously, $[N_1, \langle a\rangle] = N_{i_1} \times \cdots \times N_{i_k} = [N_1, \langle b\rangle]$. Therefore $[N_1, \langle a\rangle] = [N_1, \langle b\rangle] \leq [N_1, \tilde{B}_\pi] \cap [N_1, A_\pi]$. Now from Step 8 we have that

$$[[N_1, \tilde{B}_\pi], [N_1, A_\pi]] \leq [A_\pi^K, \tilde{B}_\pi^K] = 1$$

and so $N_1, N_{i_2}, \dots, N_{i_k}$ are abelian, which is a contradiction. The assertion is now proved.

Step 10. $G = AN = BN = AB$.

Assume that this is not true and, for instance, $AN < G$. Then we can apply Step 8 with $M = N$. In particular, $[A_\pi^K, \tilde{B}_\pi^K] = 1$, where $K = AN$, $\tilde{B}_\pi = B_\pi \cap AN = B_\pi \cap A_\pi N$ and $\tilde{B}_\pi \neq 1$. Since $C_G(N) = 1$ we may assume that there exists $1 \neq b \in \tilde{B}_\pi$ such that $[N_1, \langle b\rangle] \neq 1$. But this means that $N_1 \leq [N_1, \langle b\rangle]$ and A_π centralizes this subgroup. Since A acts transitively on $\Omega = \{N_1, \dots, N_r\}$ and $A_\pi \trianglelefteq A$, it follows that A_π centralizes each N_i, for $i = 1, \dots, r$, and so $A_\pi \leq C_G(N) = 1$, a contradiction which proves that $AN = G$.

By the symmetry between A and B we can also prove $G = BN$ and we are done. $\square$

3. The soluble case with π a set of odd primes

Theorem 2. *Let π be a set of odd primes. Let the group $G = AB$ be the product of two π-decomposable soluble subgroups $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$. Then $A_\pi B_\pi = B_\pi A_\pi$ and this is a Hall π-subgroup of G.*

Proof: Assume the result is not true and let G be a counterexample of minimal order. We know by Proposition 2 that G has a unique minimal normal subgroup $N = N_1 \times \cdots \times N_r$, which is a direct product of isomorphic non-abelian simple groups $N_1, \dots, N_r$. Moreover, $G = AB = AN = BN$ and so, by Lemma 4, A and B act transitively on the set $\Omega = \{N_1, \dots, N_r\}$ and $|N_1|$ divides $|\mathrm{Out}(N_1)||N_1 \cap A||N_1 \cap B|$. Clearly $A_\pi \neq 1$, $B_\pi \neq 1$, and, moreover, $A_{\pi'} \neq 1$, $B_{\pi'} \neq 1$. Recall also that $A_{\pi'} \cap B_{\pi'} = 1$.

From [**10**] we know that N_i should be isomorphic to one of the groups in the set:

$$\mathfrak{M} = \{L_2(q),\ q > 3;\ L_3(q),\ q < 9;\ L_4(2), M_{11}, \mathrm{PS}p_4(3), U_3(8)\}.$$

We claim first that $N = N_1$ is a simple group.

We note that either $N_1 \cap A \neq 1$ or $N_1 \cap B \neq 1$ because $|N_1|$ does not divide $|\mathrm{Out}(N_1)|$. We set $\{\sigma, \sigma'\} = \{\pi, \pi'\}$. We may assume that $N_1 \cap A_\sigma \neq 1$. Then $A_{\sigma'}$ normalizes N_1. This holds also for $B_{\sigma'}$ because $A_{\sigma'}N = B_{\sigma'}N$ since $G = AN = BN$. If in addition $N_1 \cap A_{\sigma'} \neq 1$ we have also that A_σ normalizes N_1 and consequently $N = N_1$ is simple, since $G = AN$, and the claim is proved. We get analogously to the same conclusion if $N_1 \cap B_{\sigma'} \neq 1$. Let us assume now that $N_1 \cap A_{\sigma'} = 1 = N_1 \cap B_{\sigma'}$. In particular, $N_1 \cap A$ and $N_1 \cap B$ are σ-groups. On the other hand, we recall that N is not a σ-group. Hence $1 \neq |N_1|_{\sigma'}$ divides $|\mathrm{Out}(N_1)|$. We discard next this case by checking the different possibilities for N_1:

• $N_1 \in \mathfrak{M}$, $N_1 \ncong M_{11}$, $N_1 \ncong L_2(q)$, $q = p^n$. If r is a prime dividing $|\mathrm{Out}(N_1)|$, then $r \in \{2, 3\}$. But in all the considered cases $|N_1|_r > |\mathrm{Out}(N_1)|_r$ and so these are not possible cases for N_1.

• $N_1 \cong M_{11}$. This case cannot occur since $\mathrm{Out}(M_{11}) = 1$.

• $N_1 \cong L_2(q)$, $q = p^n$. From Lemma 4 we have that $N \cap A = \times_{i=1}^{r}(N_i \cap A)$, and so $N \cap A_{\sigma'} = \times_{i=1}^{r}(N_i \cap A_{\sigma'}) = 1$. Moreover, since $A_{\sigma'}$ normalizes N_1, it normalizes N_i for any $i = 1, \dots, r$, because A acts transitively on the set $\Omega = \{N_1, \dots, N_r\}$. Therefore $A_{\sigma'} \cong A_{\sigma'}N/N$ is a subgroup of $\mathrm{Out}(N_1) \times \cdots \times \mathrm{Out}(N_r)$. Analogously $B_{\sigma'} \cong B_{\sigma'}N/N$. Moreover $A_{\sigma'}N/N = B_{\sigma'}N/N$. By the structure of $\mathrm{Out}(L_2(q))$ we deduce that there exists a prime $r \in \sigma'$ such that A and B have normal Sylow r-subgroups. From Lemmas 3 and 2 we deduce that N is abelian, which is a contradiction.

Therefore our claim follows and N is a simple group.

We recall that $G = AN = BN = AB$ and so we deduce that $|N||A \cap B| = |N \cap A||N \cap B||G/N|$. In particular, if X, Y are maximal soluble subgroups of N such that $N \cap A \leq X$ and $N \cap B \leq Y$, then $|N|$ divides $|X||Y||\mathrm{Out}(N)|$. Then we will use the fact that the orders of X and Y are known from the proof of [**2**, Lemma 2.5].

We recall also that $A_\pi \neq 1$, $B_\pi \neq 1$, $A_{\pi'} \neq 1$, $B_{\pi'} \neq 1$. Moreover, we have that $|\pi(G) \cap \pi| > 1$ and $|\pi(G) \cap \pi'| > 1$ because of Lemmas 3 and 2, as N is non-abelian.

We check next that each of the possibilities for the group N leads to a contradiction.

• $N \cong L_3(3)$ and $N \cong \mathrm{PS}p_4(3)$. In both cases $|G|$ would be divided only by three distinct primes which is a contradiction.

• $N \cong M_{11}$. In this case $\mathrm{Out}(N) = 1$ and so $G = N$ is simple. Since all subgroups of the group M_{11} are known, it is easily deduced that this case cannot occur.

• $N \cong L_3(4)$ or $N \cong L_3(7)$. These cases can be excluded since, as proved in [**2**, Lemma 2.5], for these groups it is not possible that $|N|$ divides $|X||Y||\mathrm{Out}(N)|$, for soluble subgroups X and Y of N.

• $N \cong L_3(5)$. In this case $|N| = 2^5 \cdot 3 \cdot 5^3 \cdot 31$ and $|\mathrm{Out}(N)| = 2$. By [**2**, Lemma 2.5] we may suppose w.l.o.g. that $|N\cap A|$ divides $31\cdot 3$ and $|N\cap B|$ divides $2^4 \cdot 5^3$. Hence the case $G = N$ cannot occur by order arguments. So $|G/N| = 2$ and $G \cong \mathrm{Aut}(N)$. This means that $|N \cap A| = 31 \cdot 3$ and $|N \cap B| = 2^4 \cdot 5^3$. Since B is neither a π-group nor a π'-group and $2 \in \pi'$ it should be $5 \in \pi$. This fact forces the primes 3 and 31 to be in different sets of primes. But this also leads to a contradiction, since a Sylow 31-subgroup of N is self-centralizing.

• $N \cong L_3(8)$. In this case $|N| = 2^9 \cdot 3^2 \cdot 7^2 \cdot 73$ and by [**2**, Lemma 2.5] we may assume that $|N \cap A|$ divides $73 \cdot 3$ and $|N \cap B|$ divides $2^9 \cdot 7^2$. Since $|\mathrm{Out}(N)| = 2 \cdot 3$ and $|N|$ divides $|G/N||N \cap A||N \cap B|$, the cases $G = N$ and $|G/N| = 2$ are not possible by order arguments.

If either $|G/N| = 3$ or $|G/N| = 2 \cdot 3$, it follows that $|N \cap A| = 73 \cdot 3$. Since a Sylow 73-subgroup of N is self-centralizing in $\mathrm{Aut}(N)$, we can deduce that A is either a π-group or a π'-group, a contradiction.

• $N \cong L_4(2) \cong A_8$. In this case, there is no factorization $G = AB$ with A, B soluble subgroups.

• $N \cong U_3(8)$. Then $|N| = 2^9 \cdot 3^4 \cdot 7 \cdot 19$ and $|\mathrm{Out}(N)| = 2 \cdot 3^2$. By [**2**, Lemma 2.5], we may assume that $|N \cap A|$ divides $3 \cdot 19$ and $|N \cap B|$ divides $2^9 \cdot 7 \cdot 3$. Hence by order arguments it follows that $|G| \geq |N| \cdot 3^2$. Note also that since $\mathrm{Out}(N)$ is not a direct product of a 2-group and a 3-group, G/N should be a π-group or a π'-group. By [**2**, Lemma 2.5], we may assume that $|N \cap A|$ divides $3 \cdot 19$ and $|N \cap B|$ divides $2^9 \cdot 7 \cdot 3$.

If $|G/N| = 3^2$, then $|N \cap A| = 3 \cdot 19$ and $|N \cap B| = 2^9 \cdot 7 \cdot 3$. Now the fact that a Sylow 19-subgroup of N is self-centralizing in N forces 3 and 19 to belong to the same set of primes, that is, $\pi \cap \pi(G) = \{3, 19\}$ and $\pi' \cap \pi(G) = \{2, 7\}$. But then A would be a π-group, a contradiction.

Now assume that $|G/N| = 2\cdot 3^2$, that is, $G \cong \mathrm{Aut}(N)$. Then $|N\cap A| = 3 \cdot 19$, $|N \cap B| = 2^8 \cdot 7 \cdot 3$ and $2, 3$ are in the same set of primes, that is, $\pi' \cap \pi(G) = \{2, 3\}$ and $\pi \cap \pi(G) = \{7, 19\}$. But this cannot occur again because a Sylow 19-subgroup of N is self-centralizing.

• $N \cong L_2(q)$, $q = p^n$.

Recall that, in this case, $|N| = \epsilon q(q^2 - 1)$, $\epsilon = (p - 1, 2)^{-1}$, and $\mathrm{Out}(N)$ is a cyclic group of order $\epsilon^{-1}n$. From [**2**, Lemma 2.5] it follows that, apart from some exceptional cases with $q \in \{5, 7, 11, 23\}$ that we will study later, the maximal soluble subgroups X and Y of N satisfies the condition $\{X, Y\} = \{N_N(N_p), D_{\nu(q+1)}\}$, with $N_p \in \mathrm{Syl}_p(N)$, $|N_N(N_p)| = \epsilon q(q-1)$ and $D_{\nu(q+1)}$ a dihedral group of order $\nu(q+1)$ with $\nu = (2, p)$.

We claim that p does not divide $(|N \cap A|, |N \cap B|)$. Assume first that $p \in \pi$. If p would divide $(|N \cap A|, |N \cap B|)$, then $A_{\pi'} \cap N = 1 = B_{\pi'} \cap N$, since the centralizer of any element of order p in N is a p-group. Therefore $A_{\pi'} \cong A_{\pi'}N/N$ is a subgroup of $\mathrm{Out}(N)$ and, analogously, $B_{\pi'} \cong B_{\pi'}N/N$. Moreover, $A_{\pi'}N/N = B_{\pi'}N/N$. By the structure of $\mathrm{Out}(N)$ we deduce that there exists a prime $r \in \pi'$ such that A and B have normal Sylow r-subgroups. Again from Lemmas 3 and 2 we get the contradiction that N is abelian. Note that the same conclusion follows if $p \in \pi'$.

Assume, therefore, w.l.o.g. that p does not divide $|N \cap A|$. Hence we can deduce that $|N \cap B|$ divides $|N_N(N_p)| = q(q-1)/(2, q-1)$ and $|N \cap A|$ divides $|D_{\nu(q+1)}| = \nu(q+1)$. In particular, it follows that $N \cap B$ is either a π-group or a π'-group, since the centralizer of any element of order p in N is a p-group.

We claim now that p divides $|G/N|$ and, in particular, $n > 1$. Since $|N|$ divides $|G/N||N \cap A||N \cap B|$, if p does not divide $|G/N|$, it follows that $|N|_p = |N \cap B|_p$. Then a Sylow p-subgroup of $N \cap B$ is a Sylow p-subgroup of N contained in B. Hence B must be a π-group or a π'-group, because the centralizer in $\mathrm{Aut}(N)$ of any Sylow p-subgroup of N is a p-group by [**11**, 1.17], which is a contradiction.

We have that $G/N = BN/N$ and also that $|N|_p$ divides $|G/N|_p|N \cap B|_p$. Since $B_\pi \neq 1$, $B_{\pi'} \neq 1$ and $n > 1$, it is clear that there exists some outer automorphism ϕ centralizing a Sylow p-subgroup of $N \cap B$. Then it follows that $|C_N(\phi)|_p \geq |N \cap B|_p \geq q/n$. But $|C_N(\phi)|_p \leq q^{1/2}$ (see, for instance, [**5**, Chapter 12]). Hence $q \leq q^{1/2}n$, that is, $q = p^n \leq n^2$. This leads to a contradiction, except for the cases $p = 2$ and $n \leq 4$.

The case $(p, n) = (2, 3)$ can be easily excluded, since the group $L_2(2^3) = L_2(8)$ has order divisible only by three distinct primes. Finally, the case $(p, n) = (2, 4)$ is also excluded, because in this case B would be a π'-group, which is not possible.

For $q \in \{5, 7, 11, 23\}$ there exists another possibility for the maximal soluble subgroups X and Y (see [**2**, Lemma 2.5]). But note that in all these cases $G = N$ and one of the subgroups $A = N \cap A$ or $B = N \cap B$ is contained in $N_N(N_p)$ for some $N_p \in \mathrm{Syl}_p(N)$. Then A or

B should be either a π-group or a π'-group, which provides the final contradiction. $\square$

4. The soluble case with $2 \in \pi$

Theorem 3. *Let π be a set of primes with $2 \in \pi$. Let the group $G = AB$ be the product of two soluble π-decomposable subgroups $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$. Assume that the following simple groups are not involved in G:*

(i) *$L_2(2^n)$, $n \geq 2$, except if either $n = 3$ or $q = 2^n + 1 > 5$ is a Fermat prime,*

(ii) *$L_2(q)$, $q > 3$ odd, except if q is a Mersenne prime.*

Then $A_\pi B_\pi = B_\pi A_\pi$ and this is a Hall π-subgroup of G.

Proof: Assume the result is not true and let G be a couterexample of minimal order. Obviously $A_\pi \neq 1$ and $B_\pi \neq 1$. Moreover $|\pi(G) \cap \pi| > 1$ because of Lemma 3.

We can argue as in Step 1 of Proposition 2 to deduce that G has a unique minimal normal subgroup N, which is neither a π-group nor a π'-group. We note that $N = N_1 \times \cdots \times N_r$, where N_i are isomorphic non-abelian simple groups for $i = 1, \dots, r$, $C_G(N) = 1$ and $N \trianglelefteq G \leq \mathrm{Aut}(N)$.

On the other hand, we have by Theorem 2 that $A_{\pi'} B_{\pi'}$ is a Hall π'-subgroup of G. Consequently, if $A_{\pi'} \neq 1$ and $B_{\pi'} \neq 1$, it would follow from Lemma 2 the contradiction $[N, N] \leq [A_{\pi'}^G, B_{\pi'}^G] = 1$. Therefore, w.l.o.g. we may assume that $B_{\pi'} = 1$, i.e., $B = B_\pi$, and $A_{\pi'} \neq 1$. We recall that now Lemma 1 implies that the conditions $A_\pi B_\pi = B_\pi A_\pi$ and $A_\pi \leq O_\pi(G)$ are equivalent.

We claim first that $G = A_\pi N$ and N is a simple group.

The choice of G implies that $A_\pi N/N \leq T/N := O_\pi(G/N)(BN/N)$. In particular, $N \leq T = A_\pi(T \cap A_{\pi'})B$. If T were a proper subgroup of G, then $A_\pi \leq O_\pi(T) \leq C_G(N) = 1$, which is a contradiction. Consequently G/N is a π-group and, in particular, $A_{\pi'} \leq N$. Then $X := A_\pi N = A(B \cap X)$. If X were a proper subgroup of G, we would argue as above to conclude the contradiction $A_\pi \leq O_\pi(X) = 1$. Therefore $X = A_\pi N = G$.

We can deduce now that $A_{\pi'} = (N_1 \cap A_{\pi'}) \times \cdots \times (N_r \cap A_{\pi'})$ is a Hall π'-subgroup of N and A_π acts transitively by conjugacy on the components $N_1, \dots, N_r$ of N. This implies $r = 1$, that is, N is a simple group and the claim is proved.

We prove next that $G = BN$.

Assume that $NB < G$. We claim that $N = BA_{\pi'}$, $N \cap A_\pi = 1$ and $|A_\pi| = t$ for some prime t.

Let us consider $M := NB = B(NB \cap A) = BA_{\pi'}(NB \cap A_\pi)$. If we denote $R = NB \cap A_\pi$, we deduce by the choice of G that $R \leq O_\pi(M) = 1$ and, in particular, $N \cap A_\pi = 1$. Since $G = NA_\pi = (NB)A_\pi$, we deduce that $|N| = |NB|$ and so $B \leq N = BA_{\pi'}$.

Now let C be a subgroup of A_π of order t, for some prime t, and assume that $X := NC = BA_{\pi'}C$ is a proper subgroup of G. Again we deduce that $C \leq O_\pi(X) = 1$, a contradiction. Therefore, $|A_\pi| = t$ for some prime t.

Since N is a non-abelian simple group factorized as the product of two soluble subgroups of coprime orders, we have from [**10**] and [**7**, Theorem 1.1] that N should be isomorphic to one of the following: M_{11}, $L_3(3)$, $L_2(q)$ with $q > 3$ odd and $q \equiv -1(4)$, $L_2(8)$ and $L_2(2^n)$ with $2^n + 1 > 5$ a Fermat prime. (Recall that the remainder cases for $L_2(2^n)$, $n \geq 2$, are excluded by hypothesis.) We discard next all these possibilities for the group N which will show that $G = NB$.

- $N \cong M_{11}$.

We have that $A_\pi \neq 1$ is a isomorphic to a subgroup of $\mathrm{Out}(M_{11}) = 1$, a contradiction.

- $N \cong L_3(3)$.

In this case $\pi \cap \pi(G) = \{2, 3\}$ and $\pi' \cap \pi(G) = \{13\}$. Moreover the outer automorphism of order 2 of N should centralize a Sylow 13-subgroup of N but this is not true.

- $N \cong L_2(q), q > 3$ a Mersenne prime.

In this case $|\mathrm{Out}(N)| = 2$, so A_π has order 2.

The possible factorizations for N can be found in [**7**]. So we have that $\{B, A_{\pi'}\}$ should be a pair of subgroups of N among pairs of subgroups of N of type $\{N_N(N_q), D_{q+1}\}$, with $N_q \in \mathrm{Syl}_q(N)$ and D_{q+1} a dihedral group of order $q+1$. Moreover the subgroups in these pairs are maximal subgroups of N. Since $2 \in \pi$ and 2 divides $q+1$ we have $B = D_{q+1}$ and $A_{\pi'} = N_N(N_q)$; in particular $q \in \pi'$. But then it is not possible that A_π centralizes $A_{\pi'} = N_N(N_q)$, since $C_{\mathrm{Aut}(N)}(N_q)$ is a q-group by [**11**, 1.17].

- $N \cong L_2(2^n)$, for either $n = 3$ or $2^n + 1 > 5$ is a Fermat prime.

The only factorizations of $L_2(q)$, $q = 2^n$, as product of soluble subgroups of coprime orders should be among pairs of subgroups of N of type $\{N_N(N_2), C_{q+1}\}$, with C_{q+1} a cyclic group of order $q + 1$ and $N_2 \in \mathrm{Syl}_2(N)$ (see for instance [**7**]). Since $2 \in \pi$ we have $B = N_N(N_2)$ and $A_{\pi'} = C_{q+1}$. But then there exists an outer automorphism of order t in A_π centralizing the subgroup $A_{\pi'} = C_{q+1}$ which is not the case.

Now we have proved that $G = AN = BN = AB$ and so $|N||A \cap B| = |N \cap A||N \cap B||G/N|$. From now on X and Y will denote maximal soluble subgroups of N such that $N \cap A \leq X$ and $N \cap B \leq Y$, respectively, and we will use [**2**, Lemma 2.5]. We check next that each of the possibilities for the group N leads to a contradiction which will conclude the proof. Recall that we have excluded the cases $L_2(2^n)$, $n \geq 2$, except if either $n = 3$ or $r = 2^n + 1 > 5$ is a Fermat prime, and the cases $L_2(q)$, q odd, except if q is a Mersenne prime.

• $N \cong L_3(3)$. In this case $|N| = 3^3 \cdot 2^4 \cdot 13$ and $|\mathrm{Out}(N)| = 2$. Moreover, X and Y should satisfy $\{|X|, |Y|\} = \{13 \cdot 3, 3^3 \cdot 2^4\}$. By order arguments $2^3 \cdot 3^3$ divides either $|N \cap A|$ or $|N \cap B|$. Then, since a Sylow 3-subgroup of N is self-centralizing, we have $\pi \cap \pi(G) = \{2, 3\}$ and $\pi' \cap \pi(G) = \{13\}$. Moreover, since a Sylow 13-subgroup of N is also self-centralizing, the case $|N \cap A| = 13 \cdot 3$ is not possible and so $|N \cap A| = 13$. Hence the case $G = N$ cannot occur and it follows $G \cong \mathrm{Aut}(G)$. But in this case, there would exist an automorphism of N of order 2 centralizing a Sylow 13-subgroup of N, which is not possible (see [**6**]).

• $N \cong \mathrm{PS}p_4(3)$. In this case $|N| = 2^6 \cdot 3^4 \cdot 5$ and $|\mathrm{Out}(N)| = 2$. From [**2**, Lemma 2.5] it follows that $\{|X|, |Y|\} = \{2^5 \cdot 5, 3^4 \cdot 2^4\}$. By order arguments we have that 2 and 5 divides either $|N \cap A|$ or $|N \cap B|$ and 3^4 divides the other. Then $5 \in \pi$, because there are no 2-elements in N centralizing a Sylow 5-subgroup of N. Also $3 \in \pi$, since a Sylow 3-subgroup of N is self-centralizing in $\mathrm{Aut}(N)$. Consequently, G is a π-group, which is a contradiction.

• $N \cong M_{11}$. In this case $G = N$ is simple and $\{|A|, |B|\} = \{55, 2^4 \cdot 3^2\}$, which gives a contradiction with the fact that $A_\pi \neq 1$ and $A_{\pi'} \neq 1$.

• $N \cong L_3(4)$ or $N \cong L_3(7)$. These cases can be excluded as said in the proof of Theorem 2.

• $N \cong L_3(5)$. By [**2**, Lemma 2.5], one of the numbers $|N \cap A|$ and $|N \cap B|$ divides $31 \cdot 3$ and the other divides $2^4 \cdot 5^3$. Hence the case $G = N$ cannot occur by order arguments. So we may deduce that $G \cong \mathrm{Aut}(N)$ and $|G/N| = 2$. Since a Sylow 5-subgroup of N is self-centralizing in $\mathrm{Aut}(N)$, this forces the primes 2 and 5 to be in the same set of primes. Recall also that $2 \in \pi$ and B is a π-group, so we have $|N \cap B| = 2^4 \cdot 5^3$ and $|N \cap A| = 31 \cdot 3$. Since a Sylow 31-subgroup of N is self-centralizing in $\mathrm{Aut}(N)$ (see [**6**]), we deduce that A should be a π-group, which is a contradiction.

• $N \cong L_3(8)$. Now $|N| = 2^9 \cdot 3^2 \cdot 7^2 \cdot 73$, $|\mathrm{Out}(N)| = 2 \cdot 3$ and from [**2**, Lemma 2.5] it follows that one of the numbers $|N \cap A|$ and $|N \cap B|$ divides $73 \cdot 3$, and the other divides $2^9 \cdot 7^2$. The cases $G = N$ and $|G/N| = 2$ cannot occur by order arguments. Moreover, since G/N is a π-group, we have $\{2, 3\} \subseteq \pi$. The fact that B is a π-group and a Sylow 73-subgroup of N is self-centralizing forces that $\pi = \{2, 3, 73\}$ and $\pi' = \{7\}$. The case $|G/N| = 3$ and $|N \cap A| = 2^9 \cdot 7^2$ cannot occur since a Sylow 2-subgroup of N is self-centralizing. So, $|G/N| = 2 \cdot 3$ and $|N \cap A| = 2^8 \cdot 7^2$. But in this case $N \cap A$ would be a normal subgroup of a Borel subgroup of N containing a central subgroup of order 7^2 which is a contradiction.

• $N \cong L_4(2) \cong A_8$. This case is not possible because there is no factorization of G with soluble factors.

• $N \cong U_3(8)$. Recall that $|N| = 2^9 \cdot 3^4 \cdot 7 \cdot 19$, $|\mathrm{Out}(N)| = 2 \cdot 3^2$ and by [**2**, Lemma 2.5], it should be $|G| \geq |N| \cdot 3^2$. Moreover, G/N is a π-group and $\{2, 3\} \subseteq \pi$.

If $|G/N| = 3^2$, then $\{|N \cap A|, |N \cap B|\} = \{3 \cdot 19, 2^9 \cdot 7 \cdot 3\}$, and so the fact that a Sylow 19-subgroup is self-centralizing in N leads to $\pi \cap \pi(G) = \{2, 3, 19\}$. But if $\pi' \cap \pi(G) = \{7\}$, there would be an element of order 7 in N centralizing a Sylow 2-subgroup of N, a contradiction.

Now assume that $|G/N| = 2 \cdot 3^2$ and so $\{|N \cap A|, |N \cap B|\} = \{3 \cdot 19, 2^8 \cdot 7 \cdot 3\}$ or $\{|N \cap A|, |N \cap B|\} = \{3 \cdot 19, 2^9 \cdot 7 \cdot 3\}$. In any case it follows $19 \in \pi$, since a Sylow 19-subgroup of N is self-centralizing. But $\pi' \cap \pi(G) = \{7\}$ cannot occur again because this would mean in both cases that a Borel subgroup of N would have a subgroup of order 7 centralizing a subgroup of order 2^8, which is not possible.

• $N \cong L_2(q)$, $q > 3$ a Mersenne prime.

In this case, we know from [**2**, Lemma 2.5] that $|\mathrm{Out}(N)| = 2$ and $\{X, Y\} = \{N_N(N_q), D_{q+1}\}$, with $N_q \in \mathrm{Syl}_q(N)$ and D_{q+1} a dihedral group of order $q + 1 = 2^n$, for some $n \geq 2$. (For $q = 2^3 - 1 = 7$ there exist another factorization which will be considered later.)

Since D_{q+1} is a 2-group, it follows that $N \cap A \subseteq N_N(N_q)$. Now by order arguments q divides $|N \cap A|$. Since a Sylow q-subgroup of N is self-centralizing in $\mathrm{Aut}(N)$, we deduce that A is either a π-group or a π'-group which is a contradiction.

If $q = 7$, it might be also possible that $\{X, Y\} = \{N_N(N_q), S_4\}$ with $N_q \in \mathrm{Syl}_q(N)$ and S_4 the symmetric group of degree 4. Since N_q is self-centralizing in $\mathrm{Aut}(N)$, we deduce that $N \cap B \subseteq N_N(N_q)$ and $N \cap A \subseteq S_4$. Then the factorization $A = A_\pi \times A_{\pi'}$ with $A_{\pi'} \neq 1$ and $A_\pi \neq 1$ is not possible.

• $N \cong L_2(2^n)$, for either $n = 3$ or $2^n + 1 > 5$ a Fermat prime.

Set $q = 2^n$. Recall that, in this case, $|N| = q(q^2 - 1)$, and $\mathrm{Out}(N)$ is a cyclic group of order n. From [**2**, Lemma 2.5] it follows that $\{X, Y\} = \{N_N(N_2), D_{2(q+1)}\}$, with $N_2 \in \mathrm{Syl}_2(N)$, $|N_N(N_2)| = q(q-1)$ and $D_{2(q+1)}$ a dihedral group of order $2(q + 1)$. Since the subgroups of prime order $q + 1$ in N are self-centralizing in $\mathrm{Aut}(N)$ and $q + 1$ does not divide $|\mathrm{Out}(N)|$, we deduce that $N \cap A \not\leq D_{2(q+1)}$. Hence $N \cap A \leq N_N(N_2)$. But again the fact that a Sylow 2-subgroup of N is self-centralizing in $\mathrm{Aut}(N)$ provides the final contradiction. □

Remark. In [**12**, Final examples, 3] it has been shown that the conclusion of Theorem 3 is not true for the groups $L_2(2^n)$, $n \geq 2$, except if either $n = 3$ or $2^n + 1$ is a Fermat prime.

Next we show that Theorem 3 is also false for groups involving $L_2(q)$, $q > 3$ odd, except if q is a Mersenne prime. (We note that $L_2(4) \cong L_2(5)$.) To see this we consider the group $G = PGL_2(q)$, q odd. Note that $|G : L_2(q)| = 2$. Thus $|G| = q(q^2 - 1)$ and it is known that this group has cyclic subgroups of orders $(q - 1)$ and $(q + 1)$. Then $G = AB$ where $A \cong C_{q+1}$ is a cyclic group of order $q + 1$ and $B = N_G(G_p)$, $G_p \in \mathrm{Syl}_p(G)$, is a subgroup of order $q(q-1)$. Clearly $\pi(A) \cap \pi(B) = \{2\}$. Set $\pi = \pi(N_G(G_p))$ and note that $2 \in \pi$. Then $A = O_\pi(A) \times O_{\pi'}(A)$ is a π-decomposable group and B is a π-group, but $O_\pi(A)B$ is not a subgroup, except if $q + 1$ is a power of 2, that is, q is a Mersenne prime, in which case G is a π-group.

As a consequence of Theorems 2 and 3 we deduce the following result for an arbitrary set of primes π.

Corollary 1. *Let π be a set of primes. Let the group $G = AB$ be the product of two soluble π-decomposable subgroups $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$. Assume that the following simple groups are not involved in G:*

(i) *$L_2(2^n)$, $n \geq 2$, except if either $n = 3$ or $q = 2^n + 1 > 5$ is a Fermat prime,*

(ii) *$L_2(q)$, q odd, except if q is a Mersenne prime.*

Then the composition factors of G belong to one of the following types:

(1) *π-groups,*

(2) *π'-groups,*

(3) *the following groups in the list of Fisman* [**7**, Theorem 1.1]*:*

(i) *$L_2(2^n)$, $n \geq 2$, with either $n = 3$ or $q = 2^n+1 > 5$ is a Fermat prime,*

(ii) *$L_2(q)$ with $q > 3$ and q is a Mersenne prime,*

(iii) *$L_3(3)$,*

(iv) *M_{11}.*

In particular, let the group $G = AB$ be the product of the two soluble π-decomposable subgroups $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$ and assume that the simple groups $L_2(q)$, $q > 3$, $L_3(3)$ and M_{11} are not involved in G. Then the group G is π-separable.

Proof: The last statement of the corollary follows directly from the first part. Assume that this one is not true and let G be a counterexample of minimal order. Since G/M satisfies the corresponding hypotheses for each normal subgroup M, we may assume that G has a unique minimal normal subgroup, say N. We can also deduce that $O_{\pi'}(G) = O_\pi(G) = 1$, and so N is non-abelian. Assume, for instance, that $2 \in \pi'$. From Theorem 2 we have that $A_\pi B_\pi = B_\pi A_\pi$ and, by Lemma 2, we deduce that $[A_\pi^G, B_\pi^G] = 1$, which is a contradiction to the fact that N is non-abelian, unless either $A_\pi = 1$ or $B_\pi = 1$. Now applying Theorem 3 in a similar way we deduce that either $A_{\pi'} = 1$ or $B_{\pi'} = 1$. Then, in any of the cases, G would be the product of a π-group and a π'-group and the conclusion follows from [**7**, Theorem 1.1]. $\square$

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